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B.M.S. COLLEGE FOR WOMEN

BENGALURU -560004

III SEMESTER END EXAMINATION – APRIL - 2024

M.Sc. MATHEMATICS - FUNCTIONAL ANALYSIS
(CBCS Scheme – F+R)

Course Code: MM302T **QP Code: 13002**
Duration: 3 Hours **Max. Marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. a) Show that a l_2^n space is a Banach space.

b) Show that the collection $B(X)$ of all bounded functions on a set X is a normed linear space.

(10 + 4)

2. a) Show that in a normed linear space

(i) the norm is continuous.

(ii) addition is jointly continuous

(iii) scalar multiplication is continuous.

b) Let N be a non-zero normed linear space. Show that N is a Banach space if and only if $S = \{x \in N: \| x \| = 1\}$ is complete. **(6 + 8)**

3. a) Let M be a closed linear subspace of a normed linear space N. Show that $\frac{N}{M}$ is a normed linear space with the norm defined by $\| x + M \| = inf\ \{\| x + m \|: m \in M\}$. Also if N is a Banach space then prove that $\frac{N}{M}$ is also a Banach space.

b) If $S: N \to N'$ is a continuous linear transformation and M a null space, then show that S induces a natural linear transformation S' of $\frac{N}{M}$ to N' such that $\| S \| = \| S' \|$. **(8 + 6)**

4. State and prove Hahn-Banach theorem for a complex normed linear space. **(14)**

5. a) Define an inner product space and show that

i) $| < x, y > \leq \| x \| \| y \|$

ii) $\| x + y \| \leq \| x \| + \| y \|$.

b) Prove that in a Hilbert space H translation preserves convexity.

c) Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **(5 + 3 + 6)**

6. a) Define an orthonormal set and show that if $\{e_1, e_2, \ldots, e_n\}$ is a finite orthonormal set in a Hilbert Space H, then

i) for an arbitrary $x \in H$, $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq \|x\|^2$,

ii) $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j \quad \forall j$.

b) State and prove Riesz Representation theorem for continuous linear functionals on a Hilbert space. **(7 + 7)**

7. a) If 0 and I are zero and identity operators on H respectively, then prove that $0^* = 0$ and $I^* = I$. Also show that if T is a non – singular operator on H then T^* is also non – singular.

b) Let T be an operator on a Hilbert space H. Then prove the following

i) $T = 0$ if and only if $< Tx, y > = 0, \forall x, y \in H$.

ii) $T = 0$ if and only if $< Tx, x > = 0, \forall x \in H$.

c) Show that an operator N on a Hilbert space H is normal if and only if $\| Nx \| = \| N^* x \|$ for every $x \in H$. **(4 + 6 + 4)**

8. a) Prove that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

b) State and prove spectral theorem. **(4 + 10)**

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